Delta Functions

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Two distinct (but similar) mathematical entities exist both of which are sometimes referred to as the "Delta Function." You should be aware of what both of them do and how they differ. One is called the Dirac Delta function, the other the Kronecker Delta. In practice, both the Dirac and Kronecker delta functions are used to "select" the value of a function of interest, $f(x)$ at some specific location in the respective function's domain (i.e. to evaluate $f(x)$ at some point $x = x_0$). This happens by placing $f(x)$ next to the appropriate delta function inside of an an integral (Dirac) or within a summation (Kronecker). Mathematically:

$$f(x_0) = \int_{-\infty}^{\infty} dx\, \delta(x - x_0) f(x) \tag{1}$$

$$a_n = \sum_i \delta_{i,n} a_i \tag{2}$$

1 Kronecker Delta

The Kronecker Delta $\delta_{i,j}$ is a function of the 2 arguments i and j. If i and j are the same value (i.e. $i = j$) then the function $\delta_{i,j}$ is equal to 1. Otherwise the Kronecker Delta is equal to zero. Formally this is written:

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{3}$$

So for example $\delta_{1,1} = \delta_{-1,-1} = \delta_{2006,2006} = 1$, while $\delta_{0,1} = \delta_{-1,1} = \delta_{1,27} = 0$. Get it? Don't forget of course that the variables i and j don't always have to be specifically the letters i and j. They could be m and n or whatever letters the author likes. Furthermore, some authors prefer to leave out the comma entirely, i.e.

$$\delta_{i,j} = \delta_{ij}$$

So what is this thing used for? It most often appears inside of a summation next to a vector. Say you had the vector: $\vec{a} = (4, 5, 6)$. If you wanted to simply total the components of the vector, you could write:

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3 = 4 + 5 + 6 = 15$$

Great. Just for kicks let's investigate the action of the Kronecker Delta on this expression. If we stick a Kronecker Delta next to the vector in the sum and set $j = 2$ we get:

$$\sum_{i=1}^{3} a_i \delta_{i,2} = a_1 \delta_{1,2} + a_2 \delta_{2,2} + a_3 \delta_{3,2} = 4 \times 0 + 5 \times 1 + 6 \times 0 = 5$$

Don't forget, the index i is called a "dummy variable" meaning that it only serves to allow for indexing in the sum. So I could change all of the i's to n's and the result would be the same. Equivalently I could have set $i = 2$ at the beginning and summed over j which would look like:

$$\sum_{j=1}^{3} a_j \delta_{2,j} = a_1 \delta_{2,1} + a_2 \delta_{2,2} + a_3 \delta_{2,3} = 4 \times 0 + 5 \times 1 + 6 \times 0 = 5$$

with the same result. If I wanted to really be confusing I could write the Kronecker Delta as $\delta_{j,i}$ and do the sum like:

$$\sum_{j=1}^{3} a_j \delta_{j,2} = a_1 \delta_{1,2} + a_2 \delta_{2,2} + a_3 \delta_{3,2} = 4 \times 0 + 5 \times 1 + 6 \times 0 = 5$$

The point here being that traditionally it is written $\delta_{i,j}$ but all that really matters is that you pay attention to which of the dummy variables is changing when the sum is carried out.

Exercise 1.1. How could you simplify

$$\sum_{k} \delta_{ik} \delta_{kj}$$

Finally, one actually useful application for Kronecker Delta is in doing a vector inner product. Say you had two vectors: $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. If you wanted to calculate their inner product you could write:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i b_i \tag{4}$$

But if you wanted to get a little fancier with it and use Kronecker Delta you could write:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{i,j} \tag{5}$$

If you get all of these points then you should be pretty safe with the Kronecker Delta.

2 Dirac Delta

The Dirac Delta function is a function of 1 variable, typically written $\delta(x)$. This function is defined as:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & otherwise \end{cases} \tag{6}$$

The value of x at which the delta function becomes infinite can be controlled by substituting $x - x_0$ for x so that:

$$\delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & otherwise \end{cases} \tag{7}$$

One useful way to think of the Dirac Delta function is that of the limit of a *normalized* Gaussian as the width goes to zero.

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{8}$$

Imagine (Figure: 1) taking a Gaussian and squeezing it down widthwise. Because it is normalized (total area under the curve integrates to numeric value 1) as it is squeezed in the width the height has to increase. If you make it infinitely thin then it has to be infinitely tall to compensate, but the total area is always 1.

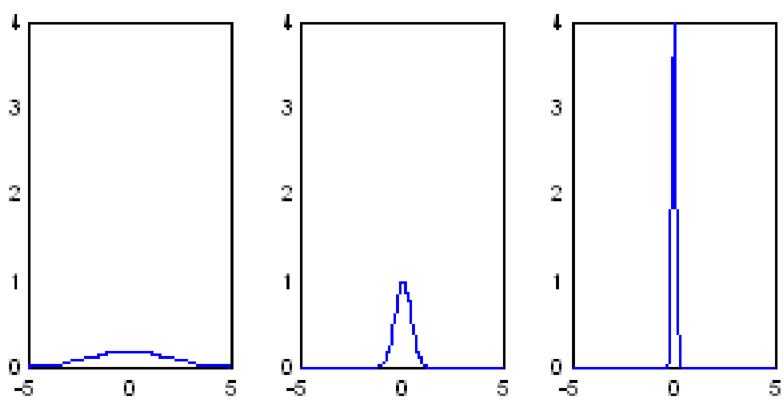


Figure 1: squeezing down a normalized gaussian

Before getting into a physical interpretation of what a Dirac Delta function might represent, let's consider some more mathematical properties of this function. Just as the Kronecker Delta usually appears inside a sum, The Dirac Delta usually appears in an integral. Perhaps the most useful definition of the delta function is:

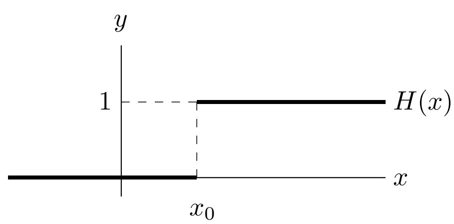
$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0) \tag{9}$$

which works for any function $f(x)$ that is continuous around $x = x_0$. This is the most common way that you will see the dirac delta function used, and really the only way to evaluate the function since infinity's really don't have physical meaning.

Exercise 2.1. Using the definition of a Dirac Delta function given in equation (9), prove that the Dirac Delta function has to be normalized. i.e. prove:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Another way that you can think of the Dirac Delta function is as the derivative of the step (Heaviside) function, $H(x)$. This function looks like:



You can see that the slope of this function is zero everywhere except at x_0. At $x = x_0$ the function makes a discontinuous vertical step, so the function's slope at this point is infinite.

This is probably enough information for you to manipulate the dirac delta function mathematically. The physical use of a Dirac Delta function will become more clear after you have done some quantum mechanics. In quantum mechanics you will use the gaussian function a lot to represent the probability distribution of some physical observable. As a physical observable becomes "very well" constrained the distribution starts to look like a Dirac Delta function. With this in mind, the Dirac Delta function is used to talk about physical observables that are arbitrarily well constrained to a certain value.

2.2 Dirac Delta Function Properties

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$