# Geometric Transform and Its Applications

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# Outline

- Geometric Transform
- Its Applications

# Homogenous system

## Rotation Matrix

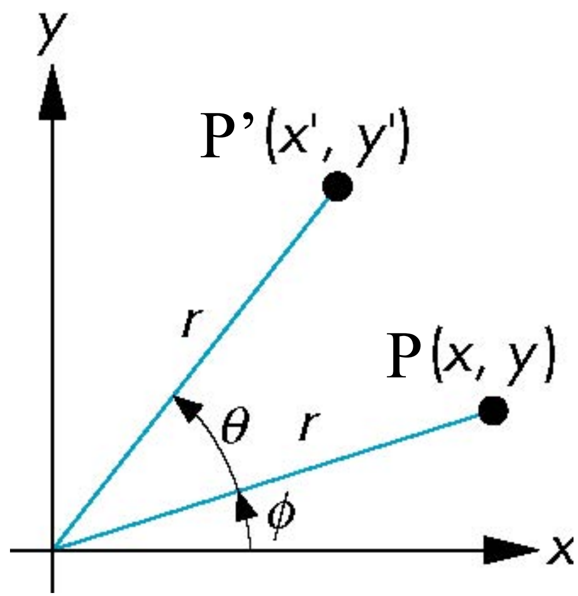
Consider Point P(x,y).

In Homogenous space:

$$x = r \cos \emptyset$$

$$y = r\, sin\, \emptyset$$

$$w = 1$$




Rotate P(x,y) an positive angle θ around the origin, to Point P'(x', y'). In Homogenous space:

$$x' = r \cos(\theta + \emptyset)$$

$$y' = r\, sin(\theta + \emptyset)$$

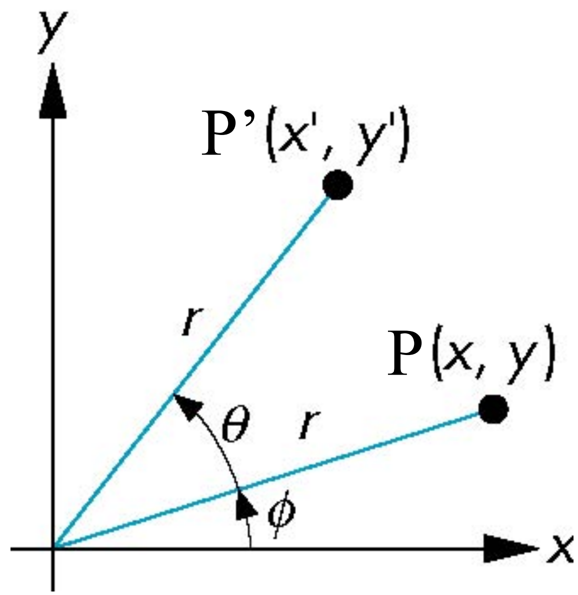
$$w' = 1$$

# Homogenous system

## Rotation Matrix

Point P'(x',y'):

$$
\begin{aligned}
x' &= r\cos(\phi+\theta) \\
&= r[\cos\phi\cos\theta - \sin\phi\sin\theta] \\
&= x\cos\theta - y\sin\theta \\
y' &= r\sin(\phi+\theta) \\
&= r[\cos\phi\sin\theta + \sin\phi\cos\theta] \\
&= x\sin\theta + y\cos\theta \\
w' &= 1
\end{aligned}
$$



# Homogenous system

## Rotation Matrix

Known form of the transform matrix:

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

And, two points P and P':

$$P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

The transformation: $P' = M.P$

$$\Rightarrow \begin{cases} x' = a_{11}x + a_{12}y + a_{13} \\ y' = a_{21}x + a_{22}y + a_{23} \\ 1 = a_{31}x + a_{32}y + a_{33} \end{cases}$$

# Homogenous system

## Rotation Matrix

Identity:

$$\begin{cases} x' & = a_{11}x + a_{12}y + a_{13} & = x\cos\theta - y\sin\theta \\ y' & = a_{21}x + a_{22}y + a_{23} & = x\sin\theta + y\cos\theta \\ 1 & = a_{31}x + a_{32}y + a_{33} & = 1 \end{cases}$$

Rotation matrix:

$$M_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system

## Translation Matrix

Identity:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x + dx \\ y' &= a_{21}x + a_{22}y + a_{23} &= y + dy \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Translation matrix:

$$M_T = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system
## Scaling Matrix

Identity:

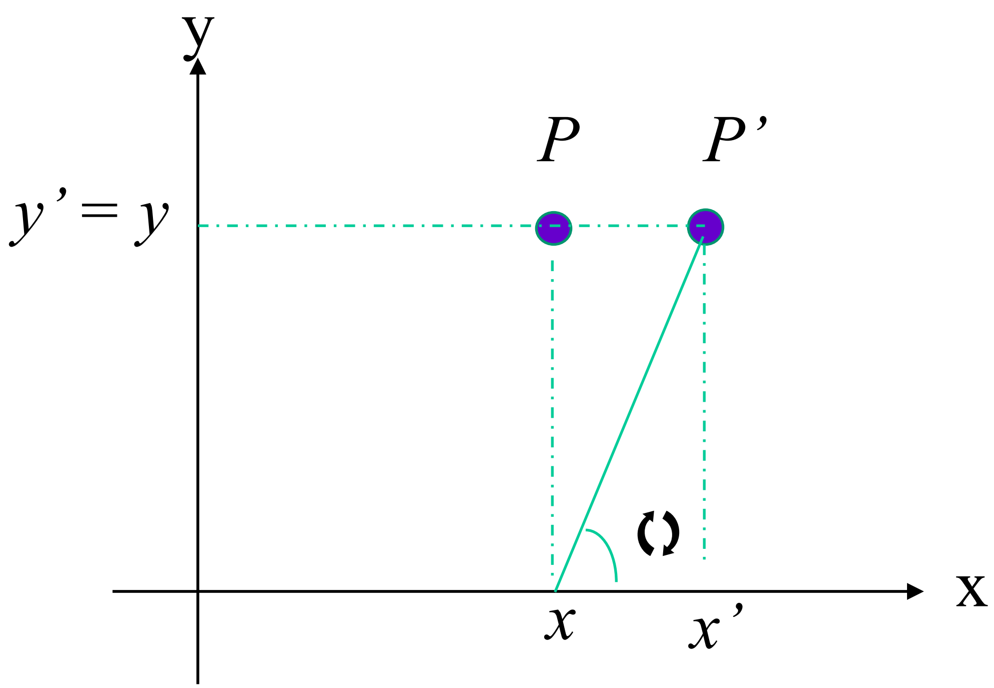
$$\begin{cases} x' & = a_{11}x + a_{12}y + a_{13} & = S_x x \\ y' & = a_{21}x + a_{22}y + a_{23} & = S_y y \\ 1 & = a_{31}x + a_{32}y + a_{33} & = 1 \end{cases}$$

Scaling matrix:

$$M_S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system

## Shearing




$$\begin{cases} x' &= x + y\cot\theta \\ y' &= y \\ w' &= 1 \end{cases}$$

# Homogenous system
## Shearing

Identity:

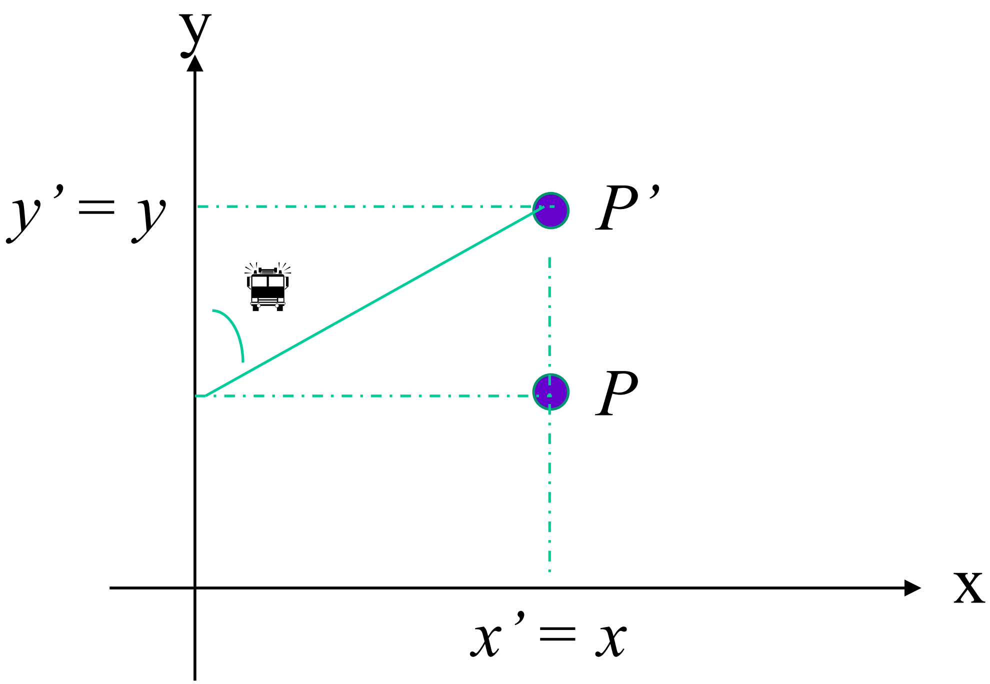
$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x + y\cot\theta \\ y' &= a_{21}x + a_{22}y + a_{23} &= y \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Shearing (along x-direction) matrix:

$$M_{SHx} = \begin{bmatrix} 1 & \cot\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system

## Shearing




$$\begin{cases} x' & = x \\ y' & = x \cot \phi + y \\ w' & = 1 \end{cases}$$

# Homogenous system

## Shearing

Identity:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_{13} &= x \\ y' &= a_{21}x + a_{22}y + a_{23} &= x\cot\phi + y \\ 1 &= a_{31}x + a_{32}y + a_{33} &= 1 \end{cases}$$

Shearing (along y-direction) matrix:

$$M_{SHy} = \begin{bmatrix} 1 & 1 & 0 \\ \cot\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system
## Shearing

Shearing (along x and y-direction) matrix:

$$M_{SH} = M_{SHy} M_{SHx}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \cot\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \cot\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cot\theta & 0 \\ \cot\phi & 1+\cot\phi\cot\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system
## Affine transform

**Scaling:**

$$M_S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Translation:**

$$M_T = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{Affine} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

**Rotation:**

$$M_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Shearing:**

$$M_{SH} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Homogenous system

## Projective transform

$$M_{\text{Projective}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \mathrm{a}_{31} & a_{32} & 1 \end{bmatrix}$$

# Projective transform Estimation

$$P' = M_{\text{Projective}} P$$

$$P' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13} \\ a_{21}x + a_{22}y + a_{23} \\ a_{31}x + a_{32}y + 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + 1} \\ \dfrac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + 1} \\ 1 \end{bmatrix}$$

# Projective transform

## Estimation

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + 1}$$

$$y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + 1}$$

$$\begin{cases} x'[a_{31}x + a_{32}y + 1] = a_{11}x + a_{12}y + a_{13} \\ y'[a_{31}x + a_{32}y + 1] = a_{21}x + a_{22}y + a_{23} \end{cases}$$

# Projective transform
# Estimation

Rearrange terms:

$$\begin{cases} a_{11}x + a_{12}y + a_{13} - a_{31}xx' - a_{32}x'y = x' \\ a_{21}x + a_{22}y + a_{23} - a_{31}xy' - a_{32}yy' = y' \end{cases}$$

Insert some dummy terms:

$$\begin{cases} a_{11}x + a_{12}y + a_{13} + a_{21}.0 + a_{22}.0 + a_{23}.0 - a_{31}xx' - a_{32}x'y = x' \\ a_{11}.0 + a_{12}.0 + a_{13}.0 + a_{21}x + a_{22}y + a_{23} - a_{31}xy' - a_{32}yy' = y' \end{cases}$$

# Projective transform Estimation

Matrix Form:

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & xx' & x'y \\ 0 & 0 & 0 & x & y & 1 & xy' & yy' \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$a_{ij}$: variables

# Projective transform Estimation

To find the value for 8 variables ($a_{ij}$), need to formulate at least 8 equations.

i.e., need at least four pairs of mapping points:

$$\langle P_1(x_1, y_1),\ P_1'(x_1', y_1') \rangle,$$
$$\langle P_2(x_2, y_2),\ P_2'(x_2', y_2') \rangle,$$
$$\langle P_3(x_3, y_3),\ P_3'(x_3', y_3') \rangle,$$
$$\langle P_4(x_4, y_4),\ P_4'(x_4', y_4') \rangle,$$

... maybe many more ...

# Projective transform Estimation

**for N pairs:**

**2N×8** **8×1** **2N×1**

Pair 1:
Pair 2:
Pair 3:
Pair 4:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x_1' & -x_1'y_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y_1' & -y_1y_1' \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x_2' & -x_2'y_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y_2' & -y_2y_2' \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x_3' & -x_3'y_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y_3' & -y_3y_3' \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x_4' & -x_4'y_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y_4' & -y_4y_4' \\ & & & & \vdots & & & \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ x_3' \\ y_3' \\ x_4' \\ y_4' \\ \vdots \end{bmatrix}$$

Additional pair:

# Projective transform Estimation

Matrix form:

$$\underset{2N\times 8}{A} \times \underset{8\times 1}{h} = \underset{2N\times 1}{b}$$

$$\underset{8\times 2N}{A^T} \; \underset{2N\times 8}{A} \times \underset{8\times 1}{h} = \underset{8\times 2N}{A^T} \; \underset{2N\times 1}{b}$$

$$\underset{8\times 8}{(A^T A)} \times \underset{8\times 1}{h} = \underset{8\times 1}{(A^T b)}$$

# Projective transform
# Estimation

Solution:

$$h = (A^T A)^{-1} A^T \ b$$

$$h = (A)^{-1} b$$

Matlab:

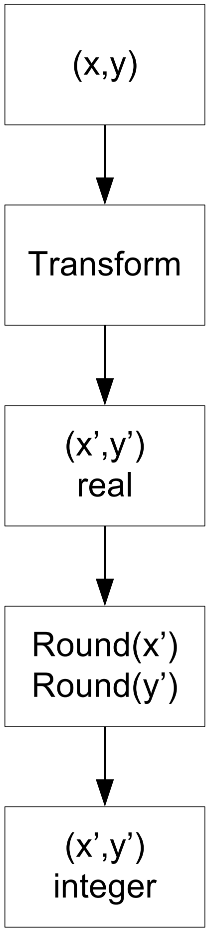
$$h = A \backslash b$$

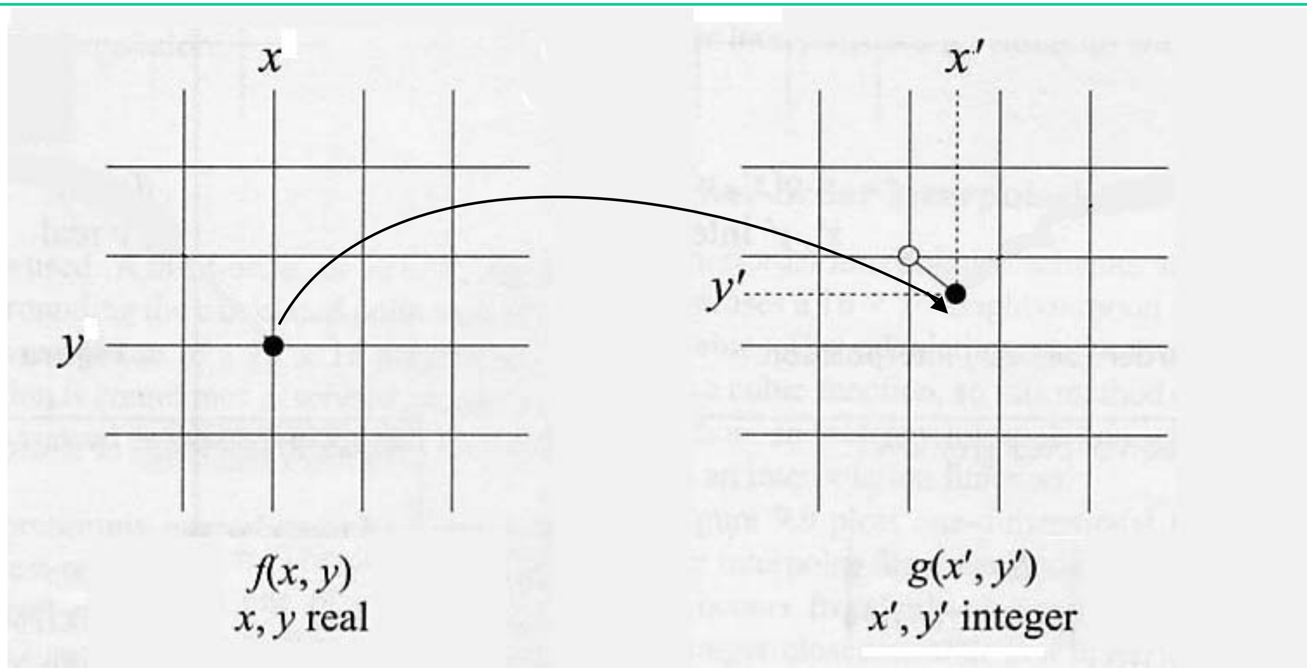
# Implementation of a transformation

- Implementation
  - Forward Mapping

    *(Holy effect)*
  - Backward Mapping
    - Zero-Order Interpolation
    - Bilinear Interpolation
    - Cubic Interpolation

# Forward mapping

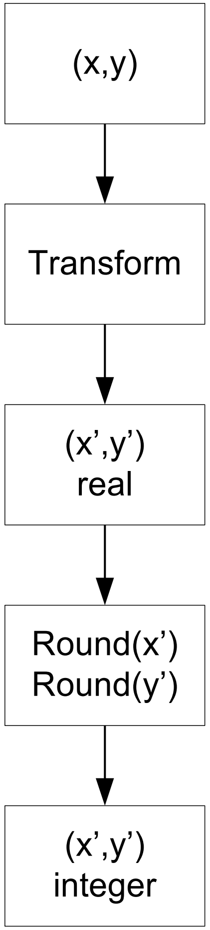


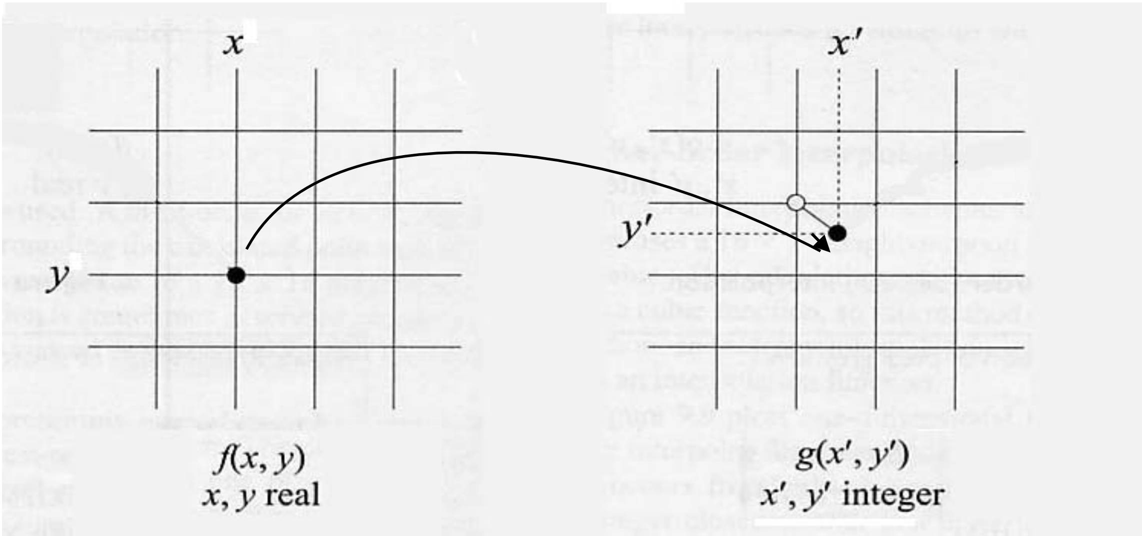




$$P' = M.P$$

$$g(round(x'), round(y')) = f(x, y)$$

# Forward mapping




**For** each pixel P

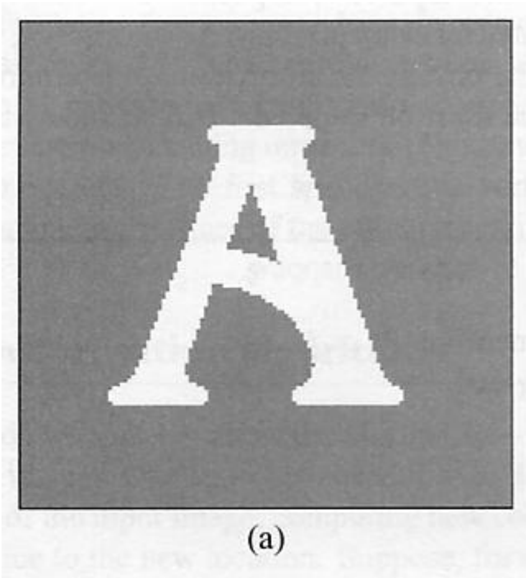
Compute $P' = M.P$

$g(round(x'), round(y'))$
$= f(x,y)$

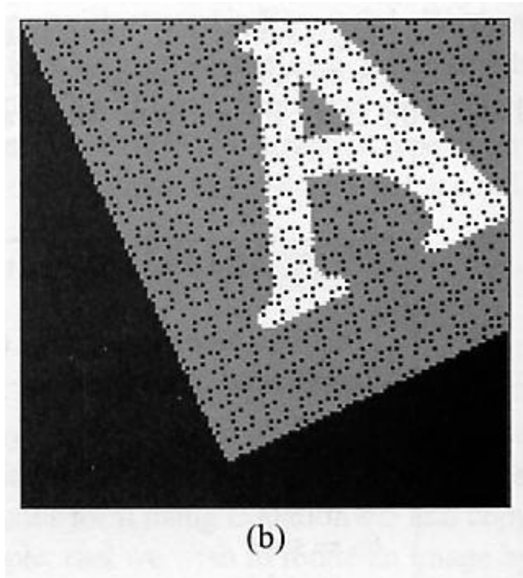
**End for**

# Forward mapping

## Example




$f(x,y)$

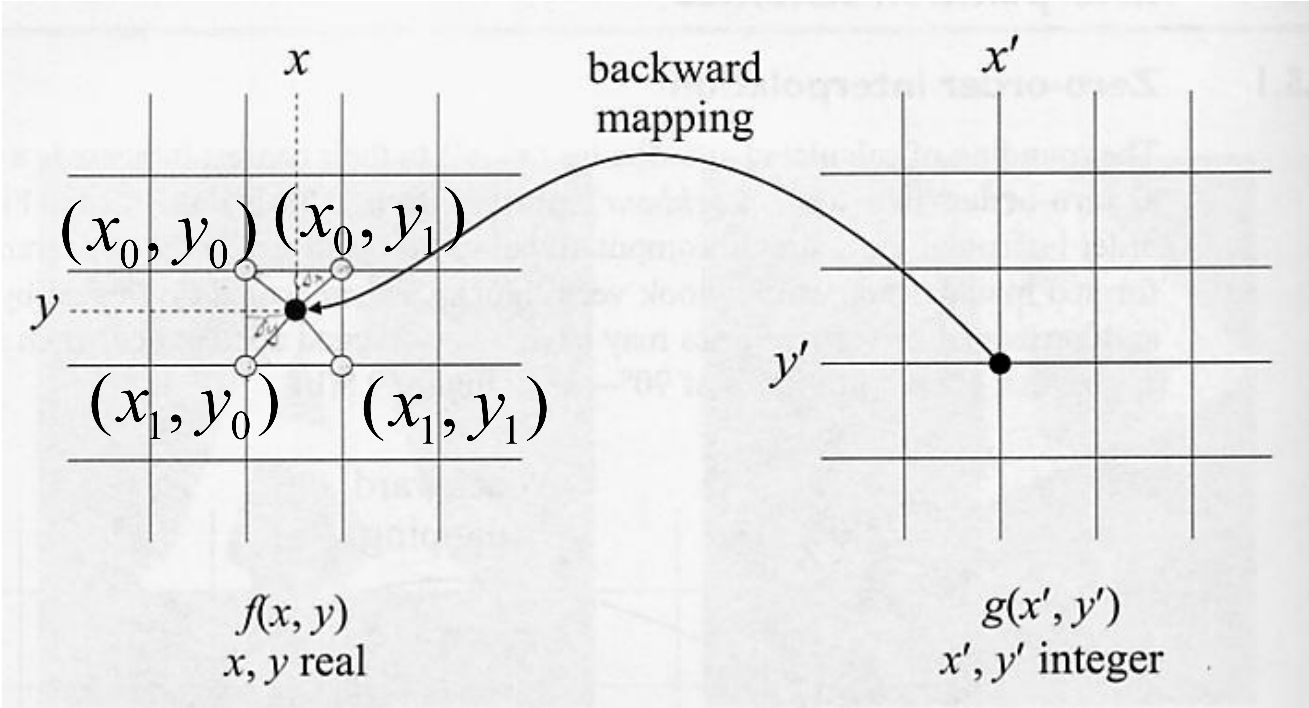



$g(x',y')$

Rotate *f(x,y)* an positive angle 25°

$$M_R = \begin{bmatrix} \cos 25^o & -\sin 25^o & 0 \\ \sin 25^o & \cos 25^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

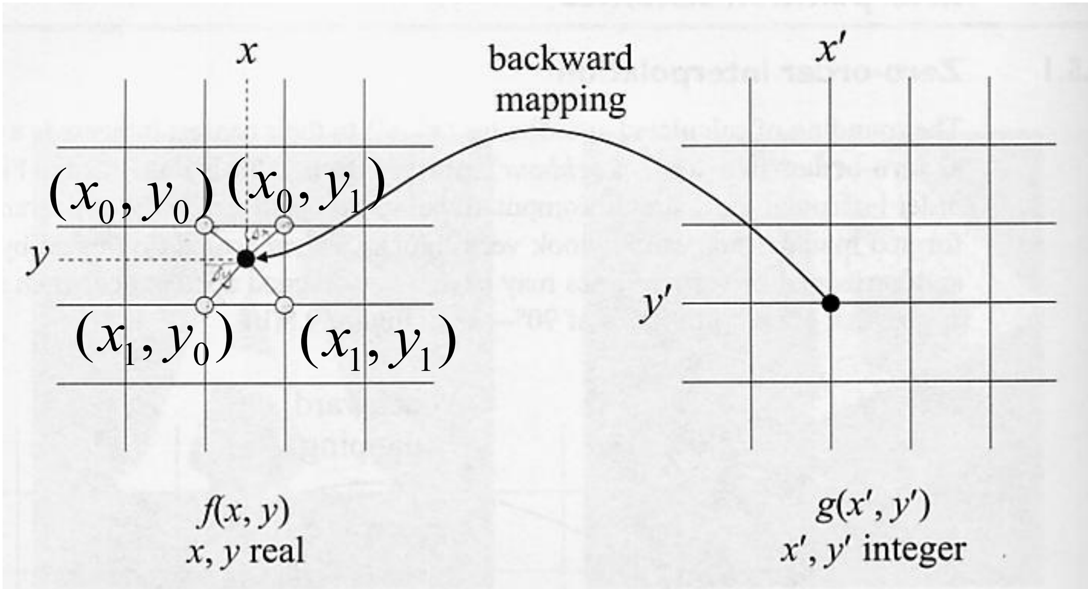
# Backward mapping




$$P = M^{-1}.P'$$

$$g(x', y') = \text{interpolation}[f(x_0, y_0)]$$

# Backward mapping




**For** each pixel P'

Compute $P= M^{-1}.P'$

$g(x', y') =$ $interpolation[f(x_o,y_o)]$

**End for**

# Backward mapping

## Inverse transform

**Scaling:**

$$M_S = \begin{bmatrix} \frac{1}{S_x} & 0 & 0 \\ 0 & \frac{1}{S_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Translation:**

$$M_T = \begin{bmatrix} 1 & 0 & -dx \\ 0 & 1 & -dy \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{Affine}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

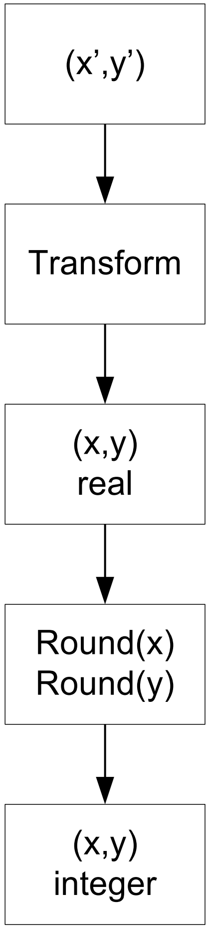
**Rotation:**

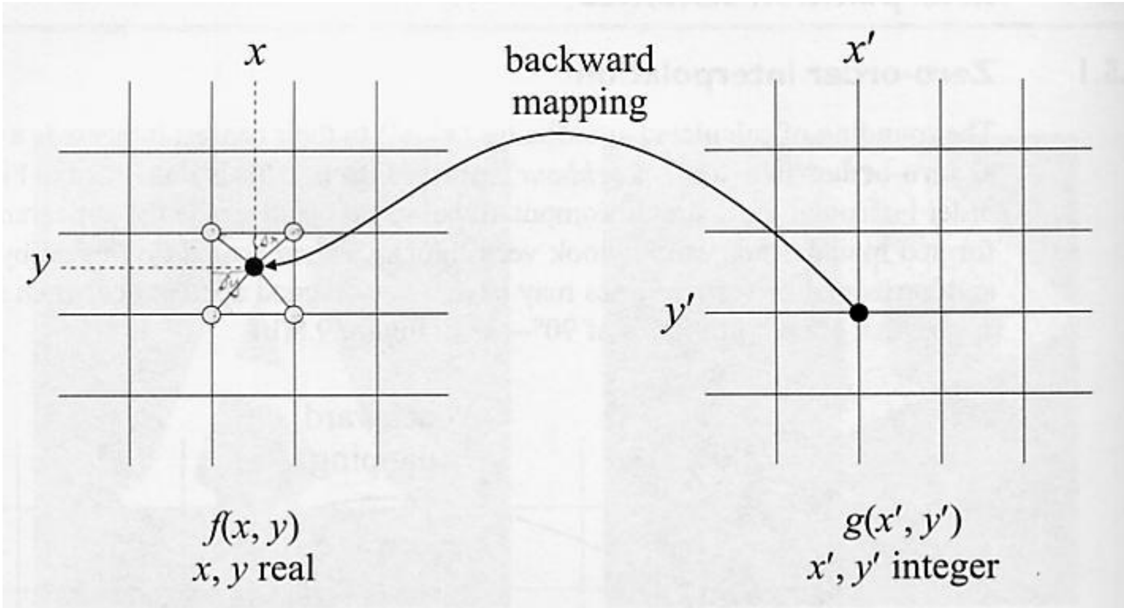
$$M_R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Shearing:**

$$M_{SH} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

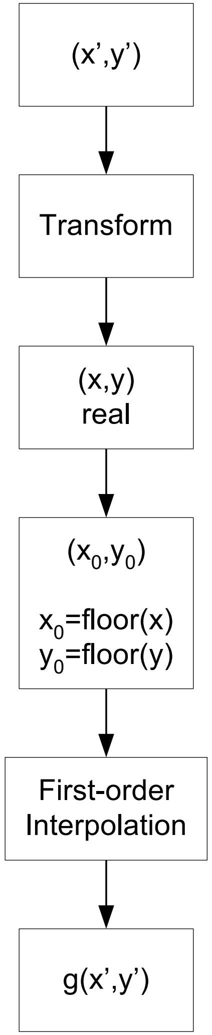
# Backward mapping
## Zero-order interpolation

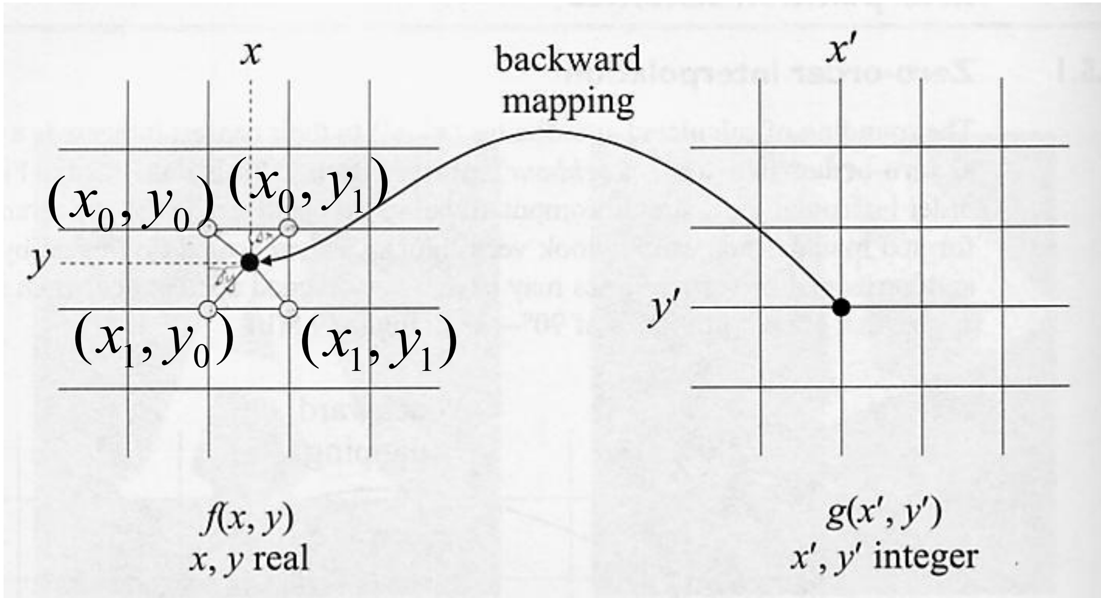



$$g(x', y') = f[round(x), round(y)]$$

# Backward mapping
## Bilinear interpolation



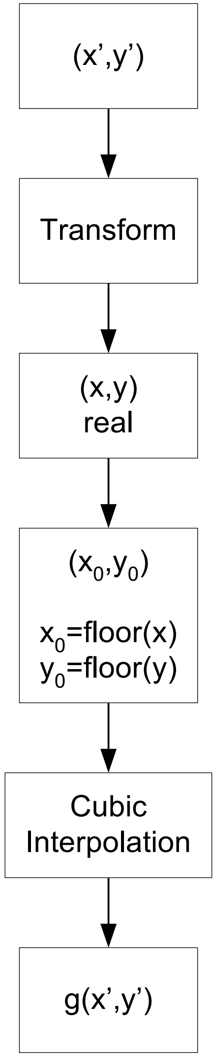



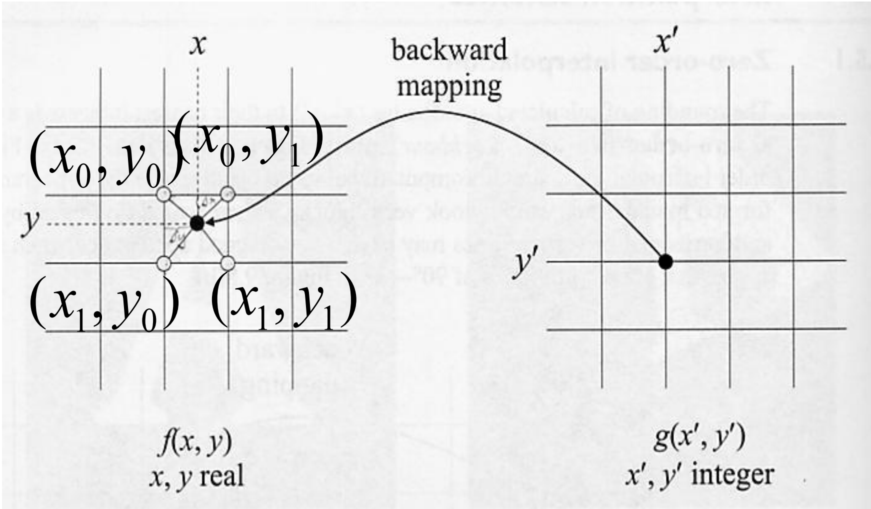

$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

$$\begin{aligned} g(x',y') = f(x_0,y_0) &+ [f(x_1,y_0) - f(x_0,y_0)]\Delta x \\ &+ [f(x_0,y_1) - f(x_0,y_0)]\Delta y \\ &+ [f(x_1,y_1) + f(x_0,y_0) - f(x_0,y_1) - f(x_1,y_0)]\Delta x \Delta y \end{aligned}$$

# Backward mapping
## Cubic interpolation




$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

$$g(x',y') = \sum_{m=-1}^{2}\sum_{n=-1}^{2} f(x_0+m, y_0+n)R(m-\Delta x)R(\Delta y - n)$$

$$R(k) = \frac{1}{6}[P(k+2)^3 - 4P(k+1)^3 - 4P(k-1)^3 + 6P(k)^3]$$

$$P(z) = \begin{cases} z & z > 0 \\ 0 & z \le 0 \end{cases}$$

# Backward mapping
## Example: zero-order interpolation

Scaling

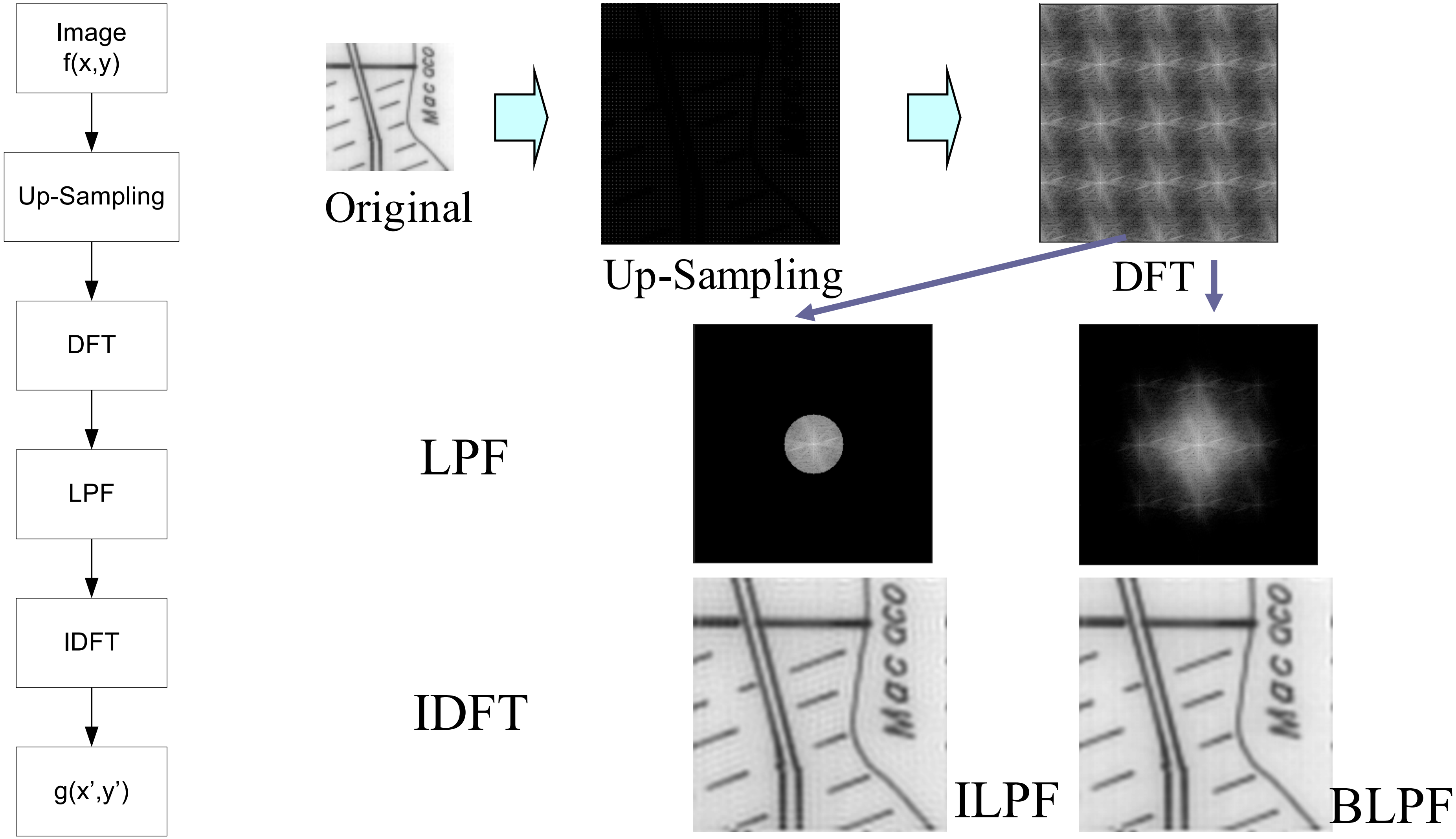
# Backward mapping
## Example: bilinear interpolation

Scaling

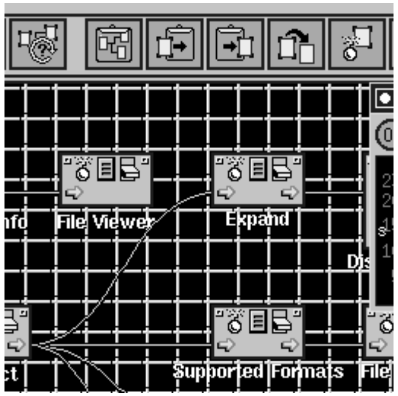
# Backward mapping
## Example: cubic interpolation

Scaling

# Processing in frequency domain



# Processing in frequency domain

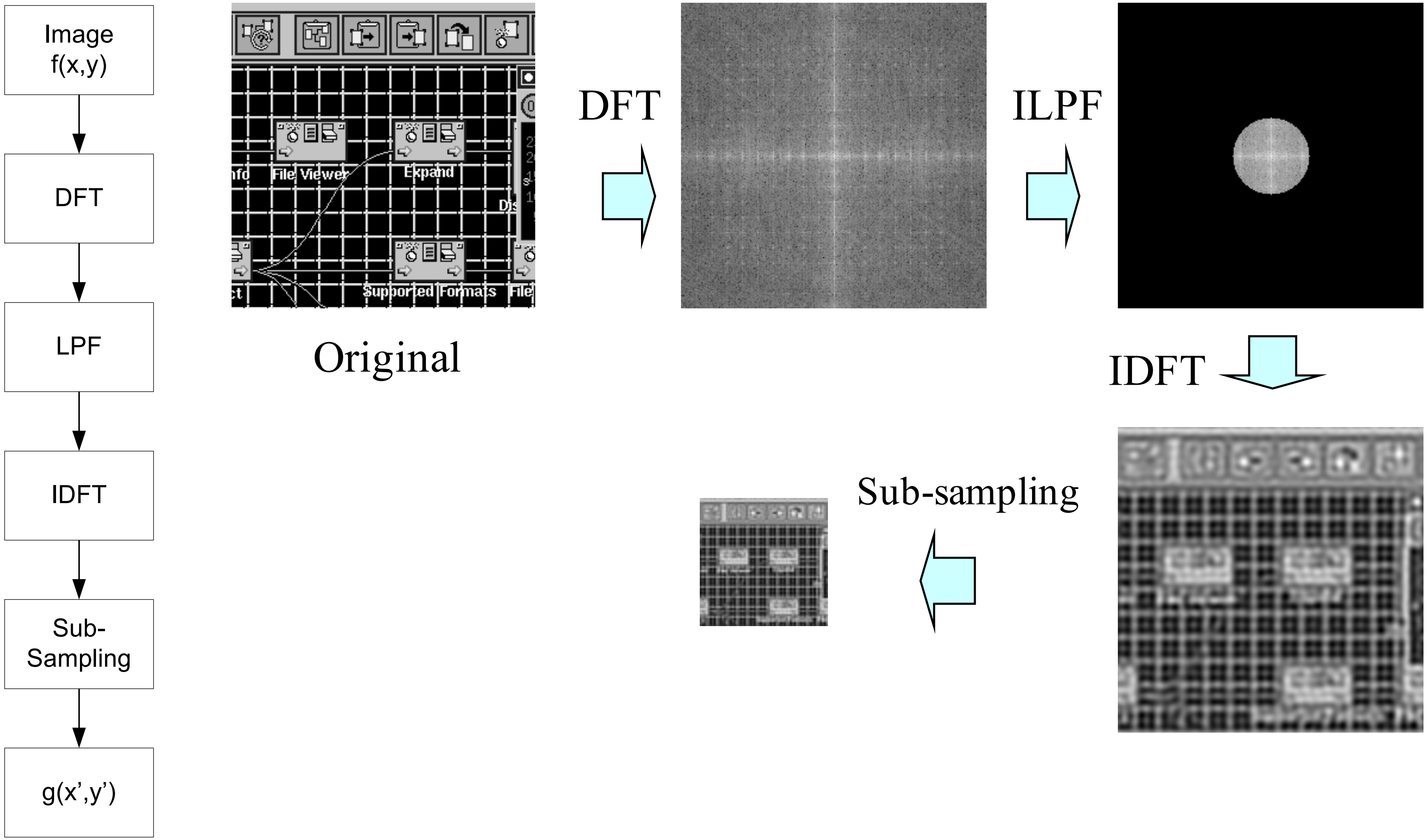



Original

Sub-sampling

Reduced Image

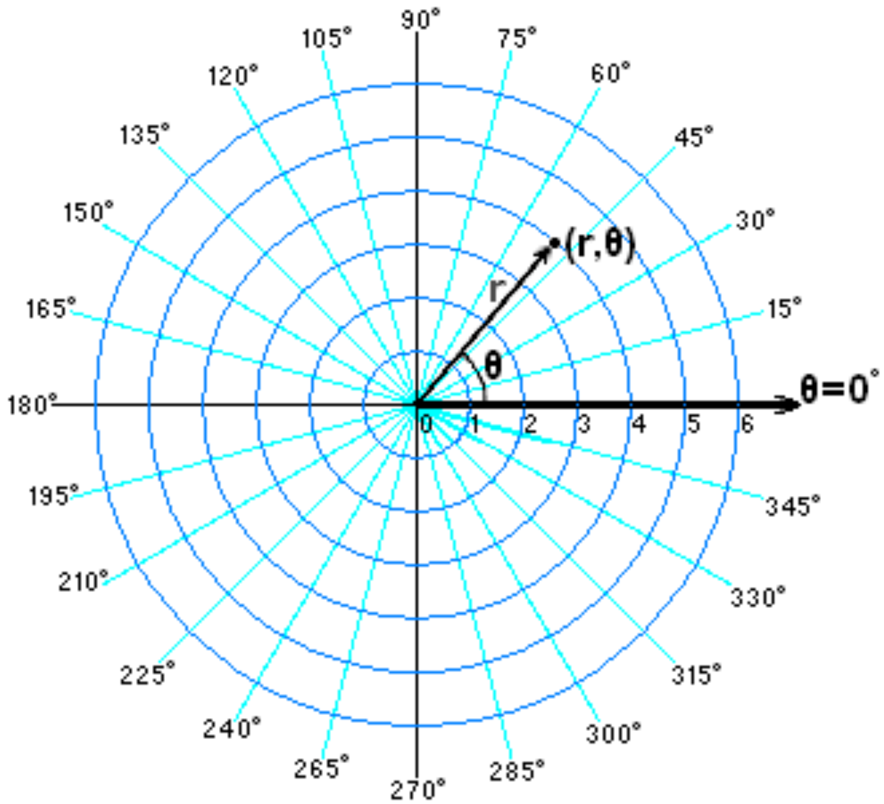
# Processing in frequency domain



# Polar Transform

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$



# Polar Transform
## Example